

FOUNDATIONAL ASPECTS OF SINGULAR INTEGRALS

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ABSTRACT. We investigate integration of classes of real-valued continuous functions on $(0,1]$. Of course difficulties arise if there is a non- L^1 element in the class, and the Hadamard finite part integral (*p.f.*) does not apply. Such singular integrals arise naturally in many contexts including PDEs and singular ODEs.

The Lebesgue integral as well as *p.f.*, starting at zero, obey two fundamental conditions: (i) they act as antiderivatives and, (ii) if $f = g$ on $(0, a)$, then their integrals from 0 to x coincide for any $x \in (0, a)$.

We find that integrals from zero with the essential properties of *p.f.*, plus positivity, exist by virtue of the Axiom of Choice (AC) on all functions on $(0, 1]$ which are $L^1((\varepsilon, 1])$ for all $\varepsilon > 0$. However, this existence proof does not provide a satisfactory construction. Without some regularity at 0, the existence of general antiderivatives which satisfy only (i) and (ii) above on classes with a non- L^1 element is independent of ZF (the usual ZFC axioms for mathematics without AC), and even of ZFDC (ZF with the Axiom of Dependent Choice). Moreover we show that there is no mathematical description that can be proved (within ZFC or even extensions of ZFC with large cardinal hypotheses) to uniquely define such an antiderivative operator.

Such results are precisely formulated for a variety of sets of functions, and proved using methods from mathematical logic, descriptive set theory and analysis. We also analyze *p.f.* on analytic functions in the punctured unit disk, and make the connection to singular initial value problems.

Keywords. Hadamard finite part, singular integrals, regularization, ZFC, Borel, Baire, independent, Borel measurable.

MSC classification numbers: 32A55, 03E15, 03E25, 03E35, 03E75.

1. INTRODUCTION

In this paper we investigate integration of classes of real-valued continuous functions on $(0,1]$ and of analytic functions in the punctured unit disk.

Integrals of functions which are singular in the interior of the interval of integration are relatively well understood. A notable example is the Hilbert transform $(Hf)(x) = \pi^{-1} \int_{-\infty}^{\infty} f(s)(s-x)^{-1} ds$ where the integrand is L^1 for large s . Evidently, the integrand is in $L^1(\mathbb{R})$ iff f vanishes identically and in general Hf needs an interpretation. Defined as a Cauchy principal value integral, the domain of H is a set of Hölder continuous functions [34]. However, by a substantial argument, regularity is *not needed* to ensure that a natural *extension* of Hf exist: H is a bounded operator on L^p for any $p \in (1, \infty)$. By a theorem of Titchmarsh, for f in L^p as above, Hf exists pointwise everywhere [49]. H also extends as a bounded operator from L^1 into weak- L^1 [43]. The extension preserves the essential properties of H .

In contrast, only sufficient conditions are known for the existence one-sided singular integrals such as

$$(1) \quad \Gamma(\alpha)J^\alpha := \int_0^x s^{\alpha-1} f(s) ds$$

with f bounded and $\operatorname{Re} \alpha < 0$. These are frequently encountered in PDEs, in the analysis of differential and pseudodifferential operators, orthogonal polynomials and many other contexts. Although the history of (1) goes back to Liouville [33] and Riemann [39], its first systematic treatment is due to Hadamard who interpreted integrals of the type $\int_a^b f(s)(b-s)^{\alpha-1} ds$ arising in hyperbolic PDEs. In (1), sufficient smoothness of f ,

$$(2) \quad f \in C^n((a, b]) \text{ and } f^{(n)}(s)s^{\alpha+n-1} \in L^1$$

allows for the Hadamard “partie finie” (*p.f.*, finite part, see. §7) to provide an extension of the usual integral. This extension, as shown by Hadamard has the properties (i) and (ii) above. In fact, it has all the properties of Lebesgue integration but positivity [20]. Riesz [38] showed that *p.f.* can be (essentially) equivalently defined by analytic continuation starting with $\operatorname{Re} \alpha > 0$; Schwartz (see [45] and [22]) reinterprets *p.f.* as a distributional regularization. See the Appendix for a brief review and further references. These reinterpretations make sense if (2) holds. In this paper, we use *Riesz’s definition of p.f.*

Related questions arise in singular ODEs, when a non- L^1 fundamental solution needs to be defined in terms of properties of solutions at the singularity. This is possible for meromorphic ODEs if the order of the poles is sufficiently low [7] or under other regularity conditions. One of the weakest such regularity condition is Écalle’s *analyzability*, see e.g. [17, 19] and [12] and references therein.

If the *integrand* in (1) is in L^1 then *p.f.* is just the Lebesgue integral. Evidently, imposing the condition that a set of functions has a non- L^1 element is *weaker* than assuming that *p.f.* is not applicable to the whole set.

In this paper we show that regularity conditions *are needed* for integrals such as (1) to make sense in sufficiently rich classes of functions with a non- L^1 element.

2. SETTING

In the following, ZFC is the standard axiomatization for mathematics. ZF is ZFC without the axiom of choice. ZFDC is ZF extended with a weak form of the axiom of choice called Dependent Choice, abbreviated as DC. In the following, $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$.¹

To analyze the existence of an inverse of differentiation \mathcal{P}_0 at a singular point, say zero, we work with functions defined to the right of zero, that is $f : (0, a] \rightarrow \mathbb{R}$ for some $a > 0$ which may depend on f . In most interesting cases, 0 will be a singular point. To avoid unwanted obstructions due to other singularities, we assume $f \in C^k((0, a])$ for some $k \in \mathbb{N} \cup \{\infty\}$. One may smoothly extend such a function to $(0, 1]$, and thus we will *assume* that all our functions are in $C((0, 1])$.

¹DC is a weak baseline form of the axiom of choice. DC is used to prove the existence of infinite sequences, and is formulated as follows. Let R be a binary relation (set of ordered pairs), where $(\forall x \in A)(\exists y \in A)(R(x, y))$. For each $x \in A$, there exists an infinite sequence $x = (x_0, x_1, \dots)$ such that $x_0 = x$ and for all $i \in \mathbb{N}$, $R(x_i, x_{i+1})$. It has been shown that DC is provably equivalent, over ZF, to the Baire category theorem for complete metric spaces. See [3].

Integration of extensions or restrictions of functions can be trivially obtained from the ones defined on functions with a common domain.

Definition 1. We define the germ equivalence relation,

$$(3) \quad f \sim_0 g \text{ if and only if } \exists a > 0 \text{ s.t. } f = g \text{ on } (0, a]$$

Definition 2. An operator A is based at 0^+ if

$$(4) \quad f \sim_0 g \Rightarrow A(f) \sim_0 A(g)$$

Operators based at other points in $\mathbb{R} \cup \{\infty\}$ are analogously defined.

2.1. Function sets. The classical normed spaces imposing size but not smoothness constraints, such as weighted L^p space and Orlicz spaces have the so called *lattice property*: if g is measurable, $|g| \leq |f|$, and f is in the space, then g is in the space.

We work in more general sets, with possible growth restrictions—through bounds—for instance as in the classical spaces above, while trying to prevent regularity at zero.² We work in structures of the type (5) below, which allow for arbitrarily fast oscillations at zero; these seem to exclude classical regularity of any kind. The following is a weakening of the lattice property.

Definition 3. For a **continuous** function f on $(0, 1]$ we denote by f^\times the multiplicative semigroup generated by f and all smooth functions bounded by 1 on $(0, 1]$:

$$(5) \quad f^\times = \{hf : h \in C^\infty((0, 1]), \|h\|_\infty \leq 1\}$$

Note 4. In this paper we use the notation f^\times **only** for functions in $C((0, 1])$.

It can be immediately verified that if $f^\times = g^\times \Rightarrow (f \in L^1 \Leftrightarrow g \in L^1)$.

2.2. Inverses of differentiation. For our negative results, we impose only the properties of the integral from zero which, arguably, any extension should satisfy.

Definition 5. For $f \in C((0, 1])$, $Op(f^\times)$ is the collection operators \mathcal{P}_0 from f^\times into $C^1((0, 1])$ s.t.

- (I) \mathcal{P}_0 is based at 0^+
- (II) $\forall g \in f^\times$ we have $(\mathcal{P}_0 g)' = g$.

The elements of $Op(f^\times)$ are clearly inverses of differentiation on f^\times . If $f \in L^1$ then $Op(f^\times) \neq \emptyset$, since the restriction of the Lebesgue integral from zero is in $Op(f^\times)$. Note that (II) implies $\mathcal{P}_0 f' = f + \text{const}$ whenever $f' \in C((0, 1])$. Also see Note 4.

Definition 6. An *extension of the integral from zero* is an operator $\mathcal{Q}_0 : C((0, 1]) \rightarrow C^1((0, 1])$ based at 0^+ such that $(\mathcal{Q}_0 f)' = f$, and with the additional properties (III) \mathcal{Q}_0 is linear, (IV) $(\mathcal{Q}_0(f))(x) = \int_0^x f$ for $f \in L^1((0, 1])$, and (V) if $f > 0$ then $\exists a$ s.t. $(\mathcal{Q}_0 f)(x) > 0$ for all $x \in (0, a)$.

EI denotes that set of all such \mathcal{Q}_0 .

Note 7. It will be clear that our negative results about an $Op(f^\times)$ are inherited by any sets (such as, say, weighted L^p spaces) containing the corresponding f^\times .

²There is no all-encompassing definition of regularity we are aware of. Certainly the integrand in (1) as a whole is not smooth in any usual sense; its *reciprocal* is smooth. An interesting discussion of smoothness and links to approximability by analytic functions, is in [16].

Note 8. The conditions in Definition 5 are not stronger than taking $AC((0, 1])$ instead of $C^1((0, 1])$ and adding “*a.e.*” after (II). Indeed, since $(\int_1^x f)' = f$ and the range of both \mathcal{P}_0 and $(x, f) \mapsto \int_1^x f$ is $AC((0, 1])$, for any f there is a constant $C(f)$ s.t. $\forall x \in (0, 1]$ we have $[\mathcal{P}_0 f](x) = \int_1^x f + C(f)$. This also implies that if $f \in C((0, 1])$, $\mathcal{P}_0 f \in C^1((0, 1])$. Then $(\mathcal{P}_0 f)' = f$ everywhere implying that for any f in the domain of \mathcal{P}_0 , all $y \in (0, 1]$ and all $x \in (0, y)$ we have $[\mathcal{P}_0(f)](y) = [\mathcal{P}_0(f)](x) + \int_x^y f$ where \int is the usual Lebesgue integral.

Note that Definition 6 imposes stronger conditions than Definition 5. That is because we are using Definition 6 mostly for positive results, and Definition 5 for the negative results.

Proposition 9. *If $f \notin L^1$ there is a $\tau \in f^\mathfrak{K}$, definable in terms of f , s.t*

$$(6) \quad \tau \geq 0 \text{ and } \int_0^1 \tau = \infty$$

The proof of this proposition is given in §4.3.

Proposition 10. *For $p \in (1, \infty)$, $L_w^p((0, 1]) \not\subseteq L^1$, iff $\int_0^1 w^{-\frac{1}{p-1}} = \infty$. If $p = \infty$ then this conclusion holds iff $\int_0^1 (1/w) = \infty$, while if $p = 1$, it holds iff $w(0) = 0$. A $\tau \in C((0, 1])$ such that (6) holds can be defined in terms of w .*

The proof of Proposition 10 is given in §4.4.

Note 11. *Existence of a \mathcal{P}_0 on a weighted L^p space, L_w^p is the same as existence of a \mathcal{P}_0 for every $f^\mathfrak{K}, f \in L_w^p$. For negative results, this allows us to only consider $Op(f^\mathfrak{K})$.*

On $C((0, 1])$ we use the topology of uniform convergence on compact sets. This topology is induced by the sequence of seminorms $\mathcal{F}(f) := (\sup_{[n^{-1}, 1]} |f|)_{n \in \mathbb{Z}^+}$. The sequence vanishes iff $f = 0$. Hence, the space is metrizable (see e.g. [28] p.3). Since the polynomials with rational coefficients are dense in this space, the space is Polish.

Definition 12. *We denote by \mathbb{P} the Polish space above. A set is said to be Borel measurable if it is Borel measurable in \mathbb{P} .*

3. MAIN RESULTS

According to Theorem 14 (a) below, ZFC proves that EI has 2^c elements. This is proved in §6.1 using the Axiom of Choice. The rest of our results are negative and establish how pathological the construction of elements of EI must be.

Specifically, according to Theorem 16, there is no description for which it is provable in ZFC that the description uniquely defines an element of $Op(f^\mathfrak{K})$ with $f \notin L^1$. In particular, this rules out usable formulas for such operators that can be proved to work within the usual axioms for mathematics.

We emphasize Theorem 16 and the sharpened form Theorem 17 over Theorem 14 (b), (c), (d). This is because the Axiom of Choice has long since transitioned from being controversial, to being accepted as part of the usual ZFC axiomatization for mathematics. However, the impossibility of giving explicit examples that can be verified to hold in ZFC represents a deeper and more serious impossibility than merely requiring the use of the axiom of choice to prove existence (beyond

the relatively benign DC). In practice, the two kinds of impossibility are closely related, although there are counterexamples to direct implications between the two. Theorems 18 and Theorem 19 provide the kind of if and only if information given by Theorem 14 (b) – but in the setting of ZFC.

In the following we denote by L^1 , by abuse of notation, the space of measurable functions $f : (0, 1] \rightarrow \mathbb{R}$ for which $\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 f$ exists.

Proposition 13. *The sets $C^k((0, 1])$, $k \in \mathbb{N} \cup \{\infty\}$ are Borel measurable subsets of \mathbb{P} .*

The proof of this proposition is given in §4.5. We say that a sentence is independent of a theory if it can neither be proved or refuted in that theory.

Theorem 14. (a) *ZFC proves that EI has 2^c elements.*

(b) *ZFDC proves the following: $Op(f^\aleph)$ has a Borel measurable element if and only if $f \in L^1$.*

(c) *ZFDC proves that, if $f \notin L^1$ and $Op(f^\aleph) \neq \emptyset$, then there is a set of reals which is not Baire measurable.*

(d) *The statement $(\exists f \notin L^1)(Op(f^\aleph) \neq \emptyset)$ is not provable in ZFDC.*

The proof of Theorem 14 (a) extends easily to the measurable functions on $(0, 1]$ for which the Lebesgue integral from 1 exists for any $\varepsilon > 0$ (the limit as $\varepsilon \rightarrow 0^+$ of the integral may not exist).

Also, note the important if and only if nature of Theorem 14(b).

The following is an easy corollary of Theorem 14.

Theorem 15. *EI $\neq \emptyset$ is independent of ZFDC. For any $k \in \mathbb{N} \cup \{\infty\}$, $(\exists f \in C^k((0, 1]) \setminus L^1)(Op(f^\aleph) \neq \emptyset)$ is also independent of ZFDC.*

Theorem 16. *There is no definition which, provably in ZFC, uniquely defines some element of some $Op(f^\aleph)$ with $f \notin L^1$. This also holds for ZFC extended by the usual large cardinal hypotheses.*

Theorem 17. *There is no definition which, provably in ZFC, uniquely defines a function whose domain is a set of real numbers with a value that is in $Op(f^\aleph)$ for some $f \notin L^1$. This also holds for ZFC extended by the usual large cardinal hypotheses.*

In order to obtain if and only if information as in Theorem 14 (b) in the context of ZFC, we need to work with concretely given $f \in C((0, 1])$. We say that $E \subset \mathbb{Z}^4$ codes an $f \in C((0, 1])$ if and only if $\forall (a, b, c, d) \in \mathbb{Z}^4$, $a/b < f(c/d)$ iff $(a, b, c, d) \in E$. An arithmetic presentation of $E \subset \mathbb{Z}^4$ takes the form $\{(a, b, c, d) \in \mathbb{Z}^4 : \phi\}$, where ϕ is a formula involving $\forall, \exists, \neg, \vee, \wedge, +, -, \cdot, <, 0, 1$, variables ranging over \mathbb{Z} , with at most the free variables a, b, c, d .

Theorem 18. *Let $E \subset \mathbb{Z}^4$ be arithmetically presented, where ZFC proves that E codes some $f \in C((0, 1])$. There is a definition, which, provably in ZFC, uniquely defines an element of $Op(f^\aleph)$ if and only if ZFC proves $f \in L^1$. This also holds for ZFC extended by the usual large cardinal hypotheses.*

The following result strengthens the forward direction in Theorem 18.

Theorem 19. *Let $E \subset \mathbb{Z}^4$ be arithmetically presented, where ZFC proves that E codes some $f \in C((0, 1])$. There is a definition which, provably in ZFC, uniquely*

defines a function whose domain is a set of real numbers, with a value in $Op(f^\aleph)$ if and only if ZFC proves $f \in L^1$. This also holds for ZFC extended by the usual large cardinal hypotheses.

Examples of arithmetically presented functions include limits of pointwise convergent sequences of rational polynomials, provided the sequence is algorithmically computable. In addition, for algorithmically computable double sequences of rational polynomials, the pointwise limit of pointwise limits is arithmetically presented, provided we have the requisite pointwise convergence. Also, compositions of arithmetically presented functions are arithmetically presented. Of course, such functions may or may not be continuous. Standard elementary and special functions over the rationals are arithmetically presented.

Note that Theorem 17 rules out any description, even using unspecified real number parameters, for which it is provable in ZFC that for some choice of these parameters, the description uniquely defines an element of $Op(f^\aleph)$, $f \notin L^1$. This rules out usable formulas for such operators that can be proved to work within the usual ZFC axioms for mathematics.

The proofs of Theorems 14 (b), (c) in §6.2 rely on the Interface Theorems from §4. The Interface Theorems show how to go explicitly from any element of $Op(f^\aleph)$, $f \notin L^1$ to a corresponding summation operator which maps $\{0, 1\}^\mathbb{N}$ into $\mathbb{R}^\mathbb{N}$. From the point of view of descriptive set theory and mathematical logic, it is easier to work with summation operators. In §5 we establish the results about summation operators that we use in §6.2, using standard techniques from descriptive set theory.

Theorem 14 (d) is proved in §6.2 and Theorem 19 is proved in §6.6. Theorem 18 is an immediate consequence of Theorem 19. Theorems 14 (d) and 19 follow from Theorem 14 (c) using well known results from mathematical logic.

The key to obtaining these negative mathematical logic results from the analytic questions is essentially the content of §4.

3.1. Links to other problems.

Note 20. Also, our results appear to preclude the existence of an integration operator over a sufficiently large class of functions defined on **No**, the surreal numbers of J.H. Conway, [9]. Indeed, if, say continuous functions extended past the gap at ∞ and an integral existed for them, then $\int_x^\infty f := F \int_x^\omega f$ where F is the finite part of a surreal number, would violate the conclusions of our theorems, since the proofs of existence of surreal objects are done using “earliness” and never use AC.

The results of Note 20 will appear elsewhere.

Note 21. For the link with *p.f.* see the Appendix.

Note 22 (Remarks about singular initial value problems). In the realm of ODEs with conditions placed at a singularity, arguably the simplest example is the linear ODE $f' = gf$ where g is singular at zero. Can conditions *at zero* separate solutions? In case g is singular at zero, but $g \in L^1((0, 1])$, then the answer is yes: the general solution is $y = (x \mapsto C + \int_0^x g)$ and $y(0) = C$ is such a condition. However, in the case $g \notin L^1$, the question reduces to the form considered in §2—by elementary operations. Many other ODEs can be brought to our setting. Such questions will be treated elsewhere.

The Hadamard *p.f.* in the complex domain is analyzed in the Appendix. The results are similar to the ones on $(0, 1]$ and we will only prove two main ones.

4. INTERFACE THEOREMS

The precise notions of summation operators at infinity are given in Definitions 24 and 25 below.

4.1. Informal description. Consider the Cantor set

$$(7) \quad \{0, 1\}^{\mathbb{N}} := \{(a_i)_{i \in \mathbb{N}} : a_i \in \{0, 1\}\}$$

For each of the sets f^{\aleph} and any nontrivial extension of integration we define a summation operator *from n to infinity* (based at infinity, in the sense of Definition 25 below, on $\{0, 1\}^{\mathbb{N}}$ with values in $\mathbb{R}^{\mathbb{N}}$. Informally, this is a finite-valued summation operator with the property that for any two sequences which coincide eventually the sum also coincides eventually (see Proposition 30):

$$(8) \quad (\exists N)(\forall n \geq N)(a_n = a'_n) \Rightarrow (\exists N)(\forall n \geq N) \left(\sum_{i=n}^{\infty} a_i = \sum_{i=n}^{\infty} a'_i \right)$$

Implausible as they might seem, such operators exist assuming AC. They are a byproduct of extensions of *p.f.* (e.g. to the whole of $C^{\infty}((0, 1])$ with *no growth or regularity condition* at zero), which also exist assuming AC. As expected, such a summation is pathological and no formula can exist for it.

To formulate negative results about the summation operator and for proving them, descriptive set theory and mathematical logic are used non-trivially.

4.2. Detailed results.

Note 23. A summation operator acting on the sequence x_n is naturally defined as a solution of the recurrence $s_{n+1} - s_n = x_n$. Clearly two solutions differ by a constant. This motivates the following.

Definition 24. Let $x \in \{0, 1\}^{\mathbb{N}}$. The standard summation for x , written $\Sigma(x)$, is $(x_0, x_0 + x_1, x_0 + x_1 + x_2, \dots)$. \mathcal{S} is a summation operator if and only if $\mathcal{S} : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$, where for all $x \in \{0, 1\}^{\mathbb{N}}$ there exists $c \in \mathbb{R}$ such that $\mathcal{S}(x) = \Sigma(x) + c$. I.e., $\mathcal{S}(x)$ is $\Sigma(x)$ with c added to all terms of $\Sigma(x)$.

Definition 25. For any set X , $X^{\mathbb{N}}$ is the set of all $f : \mathbb{N} \rightarrow X$, which is the same as the set of all infinite sequences from X indexed from 0. For $x, y \in X^{\mathbb{N}}$, $x \sim_{\infty} y$ if and only if $(\exists n)(\forall m \geq n)(x_m = y_m)$. $F : X^{\mathbb{N}} \rightarrow Y^{\mathbb{N}}$ is based at infinity if and only if for all $x, y \in X^{\mathbb{N}}$, $x \sim_{\infty} y \Rightarrow F(x) \sim_{\infty} F(y)$. We use $[x]_{\infty}$ for $\{y \in X^{\mathbb{N}} : x \sim_{\infty} y\}$.

Definition 26. Su is the collection of all summation operators on $\{0, 1\}^{\mathbb{N}}$ based at infinity.

The following two Propositions follow easily from Proposition 30 and its proof.

Proposition 27. ZF proves the following: If $f \notin L^1$ and $Op(f^{\aleph}) \neq \emptyset$ then $Su \neq \emptyset$.

Proposition 28. ZFDC³ proves the following: if there is a Borel measurable element of some $Op(f^{\aleph})$, $f \notin L^1$ then there is a Borel measurable element of Su .

³DC is used to prove basic facts about Borel measurability.

4.3. Proof of Proposition 9.

Proof. A similar construction will be used in the proof of Proposition 30. By standard measure theory, if $f \notin L^1$, then $\int f^+$ or $\int f^-$ is $+\infty$ where f^\pm are the positive/negative parts of f . Assume $\int f^+ = +\infty$ (if $\int f^- = +\infty$, the construction is essentially the same, with minor modification such as $h \leftrightarrow -h$). Define $G = \{x \in (0, 1] : f(x) \geq \frac{1}{2}\}$ and $L = \{x \in (0, 1] : f(x) \leq \frac{1}{4}\}$; they are clearly both closed in the relative topology on $(0, 1]$. Let $c_0 = \max G$ and $b_0 = \max\{x \in L : x \leq c_0\}$. Inductively, let $c_j = \max\{x \in G : x \leq b_{j-1}\}$ and $b_j = \max\{x \in L : x \leq c_j\}$. Also inductively, let $\varepsilon_j < \min\{\frac{1}{2}(c_j - b_j), \frac{1}{2}(b_j - c_{j+1})\}$ and define $h \in C^\infty((0, 1])$ be a gluing function s.t. $h = 1$ on $[b_j, c_j]$ and $h = 0$ on $[c_{j+1} + \varepsilon, b_j - \varepsilon]$. It is clear that $\tau := hf \geq 0$ and $\int_0^1 (f^+ - \tau) < 1$ where \int is the usual Lebesgue integral. \square

4.4. Proof of Proposition 10.

Proof. If $p = \infty$ then clearly $w|\tau| \leq 1$ implies $\int |\tau| = \int w|\tau|w^{-1} \leq \int_0^1 w^{-1} < \infty$ if $\int_0^1 (1/w) < \infty$. If, instead, $\int_0^1 (1/w) = \infty$ then clearly $\tau := 1/w \in L_w^\infty$ while $\int \tau = \infty$. An explicit τ with $\|\tau\|_1 = \infty$ is w^{-1} .

If $p = 1$ then we assume, without loss of generality, that $w \leq 1$. If $w(0) \neq 0$, then $w, 1/w$ are bounded, $L_w^1 = L^1$, and there is nothing to prove. Otherwise, let $A_k = \{x : w(x) \in [1/k^2, 1/(k+1)^2]\}$ and $\tau = \sum \chi_{A_k}/m(A_k)$ where m is the usual Lebesgue measure. Clearly, the A_k are disjoint and $\int_{[0,1]} \tau w \leq \sum_k k^{-2} < \infty$ while $\int_0^1 \tau = \sum_{k=1}^\infty 1 = \infty$. This τ can be made smooth as in Lemma 9.

Let $p \in (1, \infty)$ and assume $\int_0^1 w^{-\frac{p}{p-1}} < \infty$ and let $\tau \geq 0 \in L_w^p$. Then, by Hölder's inequality we have

$$(9) \quad \int_0^1 \tau = \int_0^1 [w\tau]w^{-1} \leq \left(\int_0^1 (w\tau)^p \right)^{\frac{1}{p}} \left(\int_0^1 w^{-\frac{p}{p-1}} \right)^{\frac{1}{q}} < \infty$$

Conversely, assume that $\int_0^1 w^{-\frac{p}{p-1}} = \infty$. Straightforward calculations show that $\tau(x) = w(x)^{-\frac{p}{p-1}} / \int_x^1 w^{-\frac{p}{p-1}}$ satisfies our requirements. \square

4.5. Proof of Proposition 13. J_1 , integration from 1, is continuous and injective on \mathbb{P} . Hence, by [28]⁴, for any $\mathcal{O} \subset \mathbb{P}$ open, $J_1(\mathcal{O})$ is Borel measurable. Let $D : C^1((0, 1]) \mapsto C((0, 1])$ be the usual differentiation operator. We claim that D is Borel measurable. Let $\mathcal{O} \in \mathbb{P}$ be open. Then, by elementary calculus, $f \in D^{-1}(\mathcal{O})$ iff $f \in H^{-1}(J_1(\mathcal{O}))$ where $H := f \mapsto f - f(1)$ is continuous on \mathbb{P} . By calculus, $\forall k \in \mathbb{N}$, we have $C^k((0, 1]) = (D^k)^{-1}(\mathbb{P})$. Hence $\forall k \in \mathbb{N}$, $C^k((0, 1])$ (and $C^\infty((0, 1]) = \cap_{k \in \mathbb{N}} C^k((0, 1])$) are Borel measurable.⁵ \square

Lemma 29. *Given any $f^\boxtimes, f \notin L^1$, a decreasing sequence $(\alpha_k)_{k \in \mathbb{N}}$ in $(0, 1]$ can be defined in terms of f such that*

$$(10) \quad \int_{\alpha_k}^{\alpha_{k+1}} \tau = 1$$

where τ is constructed as in Proposition 9.

⁴See Corollary 15.2, p.89.

⁵In fact, $C^k, k \in \mathbb{N} \cup \{\infty\}$ is Π_3^0 -complete, see [1] and [28], §23 D.

Proof. The function τ constructed in Proposition 9 has the property that $\theta(x) = \int_x^1 \tau$ is strictly decreasing, thus continuously invertible from \mathbb{R}^+ into \mathbb{R}^+ . We let $\alpha_0 = 1$, $\alpha_k = \theta^{-1}(k)$, $k \geq 1$ and note that (10) holds. \square

Proposition 30. *There is an explicit procedure for going from any element of $Op(f^\otimes)$, $f \notin L^1$ to an element of Su . Hence Propositions 27 and 28 hold.*

Proof. For $\tau \in T$ s.t. (6) holds, consider the open set

$$\mathcal{O}_a = \bigcup_{n: a_n=0} (\alpha_{n+1}, \alpha_n)$$

With α_k as in (10) we construct a C^∞ function h out of τ from which is 1 on \mathcal{O}_a^c and is zeroed out as in Lemma 9 with the role of (b_{n+1}, c_n) played by (α_{n+1}, α_n) dividing now ε_n by $2 \max_{(\alpha_{n+1}, \alpha_n)} \tau$ ⁶. This h has the property that the Lebesgue integral $\int_0^1 |\chi_{\mathcal{O}_a^c} \tau - \tau h| < \frac{1}{2}$. Define

$$(11) \quad x_{k;a} = [\mathcal{P}_0(\tau h)](\alpha_k) + \int_0^{\alpha_k} (\chi_{\mathcal{O}_a^c} \tau - \tau h)$$

where the last integral is the Lebesgue integral. By construction, Note 8, (8) and (11), we have $x_{n+1;a} - x_{n;a} = \int_{\alpha_{n+1}}^{\alpha_n} \tau \chi_{\mathcal{O}_a^c} = \int_{\alpha_{n+1}}^{\alpha_n} \tau = a_n$. Thus

$$\mathcal{S} = a \mapsto (x_{0;a}, x_{0;a} + x_{1;a}, x_{0;a} + x_{1;a} + x_{2;a}, \dots)$$

is a summation operator on $\{0, 1\}^\mathbb{N}$. Proposition 27 follows since all constructions have been done in ZF . Proposition 28 follows from the fact that the maps used in the constructions in this proof are manifestly Borel measurable. \square

Note 31. We remark that all the proofs in §4 are carried out in ZF .

5. SUMMATION OPERATORS

Until the proof of Theorem 39 is complete, we fix a summation operator $\mathcal{S} : \{0, 1\}^\mathbb{N} \rightarrow \mathbb{R}^\mathbb{N}$, see Definition 24, based at infinity, and prove that \mathcal{S} is not Baire measurable. We assume that \mathcal{S} is Baire measurable, and obtain a contradiction. The proof takes place within $ZFDC$, and is an application of a widely used technique from descriptive set theory. For useful information about Baire spaces and Baire category, we refer the reader to Kechris, [28], section 8.

Definition 32. *Let $f : X \rightarrow Y$, where X, Y are topological spaces, and $E \subseteq X$. We say that f is continuous over E if and only if f restricted to E is a continuous function where E is given the subspace (i.e., induced) topology.*

Lemma 33 ([28], 8.38 p. 52). *Let X be a Baire space and Y be a second countable space and assume $f : X \rightarrow Y$ is Baire measurable. Then f is continuous over a comeager subset of X .*

Lemma 34. *Let $f : X \rightarrow X$ be a bicontinuous bijection, where X is a Baire space. If $E \subseteq X$ is comeager then $f^{-1}(E)$ is comeager.*

Proof. It suffices to observe that the forward image of any dense open set under f is a dense open set. \square

⁶Nonzero by (10).

Lemma 35. *Let $E \subseteq \{0, 1\}^{\mathbb{N}}$ be comeager in $\{0, 1\}^{\mathbb{N}}$. $\{x \in \{0, 1\}^{\mathbb{N}} : [x]_{\infty} \subseteq E\}$ is comeager in $\{0, 1\}^{\mathbb{N}}$.*

Proof. We apply Lemma 34 to the Baire space $\{0, 1\}^{\mathbb{N}}$. For each nonempty finite sequence α from $\{0, 1\}$, let $\alpha^* \in \{0, 1\}^{\mathbb{N}}$ be α extended with all 0's, and f_{α} be the bicontinuous bijection of $\{0, 1\}^{\mathbb{N}}$ given by $f_{\alpha}(x) = x + \alpha^*$. Here $+$ is addition modulo 2. Obviously $\{x \in \{0, 1\}^{\mathbb{N}} : [x]_{\infty} \subseteq E\} = \cap_{\alpha} f_{\alpha}^{-1}[E]$, which by Lemma 34, is the countable intersection of sets comeager in $\{0, 1\}^{\mathbb{N}}$. \square

Lemma 36. *Let $F : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ be Baire measurable. There exists $x \in \{0, 1\}^{\mathbb{N}}$ and a finite initial segment α of x such that $(\forall y \in [x]_{\infty} \cap \{0, 1\}^{\mathbb{N}})(y \text{ extends } \alpha \Rightarrow |F(x) - F(y)| < 1)$.*

Proof. By Lemma 33, F is continuous over a comeager set $E \subseteq \{0, 1\}^{\mathbb{N}}$. By Lemma 35, fix $[x]_{\infty} \subseteq E$, and let $F(x) = c \in \mathbb{R}$. $F^{-1}[(c - \frac{1}{2}, c + \frac{1}{2})]$ is an open subset of E (as a subspace of $\{0, 1\}^{\mathbb{N}}$) that contains x . This open subset of E must contain all elements of $[x]_{\infty} \cap \{0, 1\}^{\mathbb{N}}$ that extend some particular finite initial segment α of x . \square

Definition 37. $\mathcal{S}^* : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ is defined by $\mathcal{S}^*(x) = \mathcal{S}(x) - \Sigma(x)$, which must be an element of $\mathbb{R}^{\mathbb{N}}$ whose terms are all the same. $\mathcal{S}^{**}(x)$ is the unique term of $\mathcal{S}^*(x)$.

Lemma 38. \mathcal{S}^* and \mathcal{S}^{**} are Baire measurable.

Proof. We first show that \mathcal{S}^* is Baire measurable. Let $J : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ be given by $J(x) = (\mathcal{S}(x), \Sigma(x))$. Then \mathcal{S}^* is the composition of J with subtraction; i.e., to evaluate $\mathcal{S}^*(x)$, first apply J , and then apply subtraction. Let $V \subseteq \mathbb{R}^{\mathbb{N}}$ be open. Then $(\mathcal{S}^*)^{-1}[V] = J^{-1}[W]$, where $W \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is the inverse image of subtraction on V . By the continuity of subtraction, W is open. Now W is a countable union of finite intersections of Cartesian products of open subsets of $\mathbb{R}^{\mathbb{N}}$. Note that the inverse image of J on the Cartesian product of any two open subsets of $\mathbb{R}^{\mathbb{N}}$ is Baire measurable. Hence the inverse image of J on any open subset of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is Baire measurable, as required. To see that \mathcal{S}^{**} is Baire measurable, note that \mathcal{S}^{**} is the composition of \mathcal{S}^* with the first projection function π_1 ; i.e., to evaluate $\mathcal{S}^{**}(x)$, first apply \mathcal{S}^* and then apply π_1 . Use the continuity of π_1 . \square

Theorem 39. *The following is provable in ZFDC. There is no Baire measurable summation operator $\mathcal{S} : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ based at infinity.*

Proof. We have only to complete the promised contradiction. Since \mathcal{S}^{**} is Baire measurable, by Lemma 36, fix $x \in \{0, 1\}^{\mathbb{N}}$ and a finite initial segment α of x such that $(\forall y \in [x]_{\infty} \cap \{0, 1\}^{\mathbb{N}})(y \text{ extends } \alpha \Rightarrow |\mathcal{S}^{**}(x) - \mathcal{S}^{**}(y)| < 1)$. Let $y \in [x]_{\infty} \cap \{0, 1\}^{\mathbb{N}}$ extend α and agree everywhere with x except at exactly one argument (arguments are elements of \mathbb{N}). Obviously $|\Sigma(x) - \Sigma(y)|$ is eventually 1 or eventually -1 . Since $x \sim_{\infty} y$, $\mathcal{S}(x) \sim_{\infty} \mathcal{S}(y)$, and so $\mathcal{S}(x)$ and $\mathcal{S}(y)$ eventually agree. Now $\mathcal{S}^*(x) = \mathcal{S}(x) - \Sigma(x)$ and $\mathcal{S}^*(y) = \mathcal{S}(y) - \Sigma(y)$. Hence $\mathcal{S}^*(x) - \mathcal{S}^*(y) = \mathcal{S}(x) - \mathcal{S}(y) + \Sigma(y) - \Sigma(x)$. Using the previous paragraph, $\mathcal{S}^*(x) - \mathcal{S}^*(y)$ is eventually of magnitude < 1 , $\mathcal{S}(x) - \mathcal{S}(y)$ is eventually 0, and $\Sigma(y) - \Sigma(x)$ is eventually -1 or eventually 1. We have reached the required contradiction. \square

6. PROOFS OF THE MAIN RESULTS

6.1. **Theorem 14**, (a). ZFC proves that EI has 2^c elements.

Proof. (1) Let \tilde{L} be the set of the equivalence classes induced by (3). Consider the vector space \tilde{V} generated by \tilde{L} . Let \tilde{V}_1 be the equivalence classes induced by (3) of the L^1 functions in $C((0, 1])$ and let \tilde{B}_1 be a Hamel basis in \tilde{V}_1 . By the usual construction using Zorn's Lemma let \tilde{B} be a basis for \tilde{L} containing \tilde{B}_1 .

(2) 0 will represent the equivalence class of 0. Note that any representative of \tilde{V}_1 is in L^1 .

(3) The elements b are linearly independent of each other. Indeed $\sum_{i \leq N} c_i b_i = 0$ implies $\sum_{i \leq N} c_i \tilde{b}_i \sim_0 0$, a contradiction.

(4) Let V be the vector space generated by the b 's and V_1 be the vector space generated by b_1 's (the representatives of $\tilde{b}_1 \in \tilde{V}_1$).

(5) On V_1 we let Λ be the linear functional $v_1 \rightarrow \int_0^1 v_1$. We write $V = V_1 \oplus V_2$; any v can be uniquely written as $v = v_1 + v_2, v_{1,2} \in V_{1,2}$. We let $\Lambda v = \Lambda v_1$. This is obviously a linear functional on V .

(6) Let $f \in C((0, 1])$. By assumption $f \in \tilde{f} \in \tilde{L}$ for some \tilde{f} , and \tilde{f} can be written uniquely in the form $\tilde{f} = \sum_{i=1}^N c_i \tilde{b}_i$, which is equivalent to $f = \sum_{i=1}^N c_i b_i + h, h \sim_0 0$. The decomposition is unique since

$$\sum_{i=1}^N c_i b_i + h = 0 \Leftrightarrow \sum_{i=1}^N c_i \tilde{b}_i \sim_0 0 \Leftrightarrow c_i = 0 \forall i \leq N$$

(7) Now we simply define

$$(12) \quad \mathcal{P}_0 f = \int_1^x \sum_{i=1}^N c_i b_i + \Lambda \sum_{i=1}^N c_i b_i + \int_0^x h$$

where the last integral is the Lebesgue integral, which exists since $h \sim_0 0$.

It is now straightforward to check that \mathcal{P}_0 is a linear antiderivative with the required properties. Eventual positivity comes from the fact that \mathcal{P}_0 coincides with \int_0^x in L^1 and the fact that $\int_1^x f \rightarrow -\infty$ otherwise. The property $\mathcal{P}_0 g' = g + \text{const.}$ follows from the fact that, for $x \in (0, 1]$ we have $(\mathcal{P}_0 g')(x) = (\mathcal{P}_0 g')(1) + \int_1^x g'$.

We note that we have not used continuity in this proof, and the extension claimed after the statement of Theorem 14 is obvious. \square

6.2. **Proof of Theorem 14 b.** Existence results in $Op(f^\aleph), f \in L^1$ are immediate. ZFDC proves the following. $Op(f^\aleph)$ has a Borel measurable element if and only if $f \in L^1$.

Proof. This is immediate from Proposition 28 and Theorem 39. \square

Note that the equivalence in Theorem 14 (b) does not involve provability or definability notions. Arguably, any subset of or function between Polish spaces that is not Borel measurable, is mathematically pathological or at least mathematically undesirable. The Borel measurable sets form a natural hierarchy of length the first uncountable ordinal, and it can be further argued that any subset of or function that does not lie in the first few levels of this hierarchy is pathological or

at least mathematically undesirable. Borel measurability in Polish spaces is extensively investigated in descriptive set theory, particularly in connection with Borel equivalence relations and reductions between them. See [21].

6.3. Theorem 14 c. ZFDC proves that, if $f \notin L^1$ and $Op(f^\aleph) \neq \emptyset$, then there is a set of reals which is not Baire measurable.

Proof. Assume that $Op(f^\aleph), f \notin L^1$ is nonempty. By Proposition 27, let $\mathcal{S} \in Su$. By Theorem 39, \mathcal{S} is not Baire measurable. Hence there is a subset of $\{0, 1\}^\mathbb{N}$ that is not Baire measurable in $\{0, 1\}^\mathbb{N}$. Let $T \subseteq \{0, 1\}^\mathbb{N}$ consist of removing the elements of $\{0, 1\}^\mathbb{N}$ that are eventually constant. Then T is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$. Also since we have removed only countably many points from $\{0, 1\}^\mathbb{N}$, there is a subset of T that is not Baire measurable in T . Hence there is a subset of $\mathbb{R} \setminus \mathbb{Q}$ that is not Baire measurable in $\mathbb{R} \setminus \mathbb{Q}$. Hence there is a subset of \mathbb{R} that is not Baire measurable in \mathbb{R} . \square

Lemma 40. ZFDC does not prove the existence of a set of reals that is not Baire measurable.

Proof. This is proved in [46] assuming that ZFC + “there exists a strongly inaccessible cardinal” is consistent. It was subsequently proved in [47] assuming only that ZFC is consistent. \square

6.4. Theorem 14 d. The statement $(\exists f \notin L^1)(Op(f^\aleph) \neq \emptyset)$ is not provable in ZFDC.

Proof. Suppose ZFDC proves $Op(f^\aleph) \neq \emptyset$ for an $f \notin L^1$. By Theorem 14 (c), ZFDC proves that there exists a set of reals that is not Baire measurable. This contradicts Lemma 40. \square

Theorem 15 follows from Theorem 14 a and d.

6.5. Proof of Theorems 16 and 17. The most convincing negative results of this paper are Theorems 16, 17, 18 and 19. These involve explicit definability. In many contexts in descriptive set theory, we have non Borel measurability, yet we do have demonstrably explicit definability. The most direct example of this is by constructing an $A \subseteq \mathbb{R}^2$ such that every Borel measurable $B \subseteq \mathbb{R}$ is of the form $\{y \in \mathbb{R} : (c, y) \in A\}, c \in \mathbb{R}$. Then we can form the diagonal set $\{y \in \mathbb{R} : (y, y) \notin A\}$, which obviously differs from every Borel measurable $B \subseteq \mathbb{R}$. A more mathematically interesting example is as follows. Consider the infinite product space $\mathbb{Q}^\mathbb{N}$, using the order topology on \mathbb{Q} . Then $\{x \in \mathbb{Q}^\mathbb{N} : \text{rng}(x) \text{ is a compact subset of } \mathbb{Q}\}$ is well known to be not Borel measurable.

Theorem 16 follows immediately from Theorem 17.

Lemma 41. Let T be ZFC or ZFC extended by any standard large cardinal hypothesis, such as on the Chart of Cardinals in [26]. Let M be a countable model of T . There is a countable mild forcing extension M' of M satisfying $T +$ “every set of reals is Baire measurable” in which every M' definable set of reals of M' , with reals of M' as parameters, is internally Baire measurable.

Proof. This result was proved in [46] with ZFC replaced by ZFC + “there exists a strongly inaccessible cardinal”. This result as stated was proved in the subsequent [47]. \square

Here is the formal statement of Theorem 17.

Theorem 17 (formal). *There is no formula φ of ZFC with exactly the free variables x, y , such that the following is provable in ZFC.*

- i $\varphi(x, y) \Rightarrow x \in \mathbb{R} \wedge (\exists! y)(\varphi(x, y))$.
- ii $(\exists x, y)(\varphi(x, y) \wedge (\exists f \notin L^1 \wedge y \in Op(f^\aleph)))$.

This also holds for ZFC extended by any of the usual large cardinal hypotheses, provided the extension results in a consistent system.

Proof. Let T be as in Lemma 41. Let φ be such that i,ii are provable in T . Let M, M' be as given by Lemma 41. By ii, choose $x, y \in M'$ such that $\varphi(x, y) \wedge y \in Op(y^\aleph) \wedge y \notin L^1$ holds in M' . By i, $x \in \mathbb{R}$ holds in M' , and y is M' definable from x . By Proposition 30, let $S \in Su$, where S is M' definable from y , and hence M' definable from x . By Theorem 39, S is not Baire measurable in M' . By the explicit construction in the proof of Theorem 14 (c) that converts a non Baire measurable set in $\{0, 1\}^\mathbb{N}$ to a non Baire measurable set in \mathbb{R} , we obtain a set of reals, internal to M , which is non Baire measurable in the sense of M' , and also M' definable from a real internal to M' . This contradicts Lemma 41. \square

6.6. Proof of Theorem 19. Once again, Theorem 18 follows immediately from Theorem 19. Theorem 19 is formalized analogously to Theorem 17. We omit the detailed formalization. We now prove Theorem 19, which we repeat here for the convenience of the reader.

Theorem 19. Let $E \subset \mathbb{Z}^4$ be arithmetically presented, where ZFC proves that E codes some $f \in C((0, 1])$. There is a definition which, provably in ZFC, uniquely defines a function whose domain is a set of real numbers, with a value in $Op(f^\aleph)$ if and only if ZFC proves $f \in L^1$. This also holds for ZFC extended by the usual large cardinal hypotheses.

Proof. Proof: Let E be as given. Let T be as in Lemma 41. Let T prove that E codes $f \in C((0, 1])$. Suppose T does not prove $f \in L^1$. Let M be a countable model of $T + f \notin L^1$. Let M' be as given by Lemma 41. Then M' also satisfies $f \notin L^1$. This is because $f \notin L^1$ is an arithmetic sentence. Now suppose that φ is a definition which, provably in T , uniquely defines a function whose domain is a set of real numbers, with a value in $Op(f^\aleph)$. Then in M' , we obtain an element of $Op(f^\aleph)$ that is definable from an internal real in M' . Following the proof of Theorem 17, we obtain a contradiction. \square

To prove the weaker Theorems 16 and 18, where there are no real number parameters, it suffices to use Lemma 41 with M definability without parameters. This is because the three spaces in question are explicitly defined. For this weakened form of Lemma 41, we can adhere to [46], merely generically collapsing ω_1 to ω , and weaken the assumption of the consistency of ZFC + “there exists a strongly inaccessible cardinal” to the consistency of ZFC.

7. APPENDIX: THE HADAMARD $p.f.$

In a nutshell, what is now called the Hadamard “partie finie” ($p.f.$, finite part) relies on smoothness assumptions on f to integrate by parts in (1). The infinite endpoint values are discarded at each step. This process lowers the order of the singularity of the integrand until it becomes L^1 .

As shown by M. Riesz, cf. [38], a natural way to interpret $p.f.$ is through analytic continuation with respect to α of the right side of (1), starting from a power of s for which the integrand is in L^1 . Analytic continuation from $\operatorname{Re} \alpha > 0$ to $\operatorname{Re} \alpha > -n$ exists if (2) holds. This is manifestly so: indeed after integration by parts we obtain

$$(13) \quad \Gamma(\alpha)(J^\alpha f)(x) = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha)f^{(k)}(x)}{\Gamma(\alpha+k+1)} x^{\alpha+k} + \int_0^x s^{\alpha+n-1} f^{(n)}(s) ds$$

For the distributional interpretation, see *e.g.* [22] §3.2.

This paper establishes that the above mentioned sufficient condition is also necessary in a deep sense: there is no formula further extending (13) without smoothness assumption. In this sense, the Hadamard definition is optimal: it is necessary that f have exactly the regularity required by (2), which is the same as the regularity needed for the right side of (13) to make sense.

7.1. The analytic case. Our main results can be adapted to study the $p.f.$ on meromorphic functions (where it exists) and on functions with essential singularities. The functions \mathcal{A} we study are analytic in $D = \mathbb{D} \setminus \{0\}$ and continuous up to the boundary. For analytic functions, the space f^\times cannot be adapted by simply replacing bounded C^∞ functions with bounded analytic functions on $\mathbb{D} \setminus \{0\}$, because zero would be a removable singularity, and any regularity at zero of f would be inherited by f^\times . Instead, we enlarge this space. We take a weight $0 < w \in C((0, 1])$, which we assume for simplicity to be w.t. $x^2 w(x)$ is decreasing, and define

$$w^\times := \{f \in \mathcal{A} : |f(z)| < w(|z|)\} = \mathcal{A} \cap L_w^\infty(D)$$

We require (i) $(\mathcal{P}_0(f))' = f$. However, in \mathcal{A} , \sim_0 is trivial: $f \sim_0 g \Rightarrow f \equiv g$. We impose on \mathcal{P}_0 a stronger requirement, one that is satisfied by $p.f.$ It is convenient to make the change of variables $t = 1/x$, $x^2 w(x) = W(t)$ and move the problem to infinity. We are then looking for an integral based at infinity. Note that W is increasing. We write $W(x) = x^{2g(x)-1}$; g is also increasing. Evidently, if g increases without bound, then W is superpolynomial. Let $G = g^{-1}$ and assume without loss of generality that g grows slowly enough that

$$(14) \quad \forall k \in \mathbb{Z}^+, \quad \beta_k := G(k) > 5^k$$

Our condition (ii') is “if the antiderivatives coincide on intervals, then they coincide”, more precisely

$$(15) \quad (ii') \quad f_1 \sim_\infty f_2 \text{ iff } (\forall n \in \mathbb{Z}^+ \left(\int_{\beta_{n+1}}^{\beta_n} f_1 = \int_{\beta_{n+1}}^{\beta_n} f_2 \right) \text{ then } \left((\mathcal{P}_0 f_1)(\beta_1) = (\mathcal{P}_0 f_2)(\beta_1) \right)$$

Note 42. Of course, if f_1, f_2 grow too slowly, (15) may imply $f \equiv g$; this is not a problem, as there will always be suitable sequences which do not entail $f \equiv g$. A serious problem however is that the equivalence relation depends on a non canonical $(\beta_k)_{k \in \mathbb{Z}^+}$. We leave open the question of the existence of more natural and nontrivial equivalence relations on spaces with unique continuation.

For analytic functions with an isolated singularity at zero, the classical domain of $p.f.$ is optimal:

Theorem 43. (A) ZFC proves the existence of a \mathcal{P}_0 on D with the properties (i), (ii') and also: (iii) \mathcal{P}_0 is linear, and (iv) $\mathcal{P}_0 = p.f.$ on meromorphic functions.

(B) ZFDC proves the existence of a \mathcal{P}_0 on w^\times satisfying (i) and (ii') above, iff $x^n w(x) \rightarrow 0$ as $x \rightarrow 0$ for some n .

Proof. (A) is proved as in §6.1, with small adaptations: $C((0, 1])$ is replaced by D , “ $f \in L^1$ ” by “ f is meromorphic” and \int_0^1 by $p.f.$ on $(0, 1)$; we now require in (2) that, if \tilde{V}_1 has a meromorphic function inside, its representative should be meromorphic. Finally we replace \int_0^x in (12) by $\int_{\beta_i}^x$ for any $\beta_i < x$ (cf. (15)).

For (B), if w grows at most polynomially, $p.f.$ applies. In the opposite direction, we now construct, from a \mathcal{P}_0 based at infinity, an element of Su .

Definition 44. As before $A = \{0, 1\}^\mathbb{N}$. Without loss of generality, we can restrict to the sequences with $a_1 = 0$. Let $s_k = \sum_{j=1}^k a_j, k \in \mathbb{Z}^+$. Consider the Cantor space of functions

$$(16) \quad \mathfrak{C}_1 := \left\{ F_a : F_a(z) = \sum_{k=1}^{\infty} B_k \prod_{j \neq k} \left(1 - \frac{z}{\beta_j}\right)^2, \quad a \in A \right\}; \quad B_k := \frac{s_k}{\prod_{j \neq k} (1 - \beta_k/\beta_j)^2};$$

where by construction $F_a(\beta_k) = s_k$.

Note 45. $F_a(\beta_k)$ will be our $\int_1^{\beta_k} f_a$, where $f_a = F'_a$.

By (14) we have the following straightforward estimate ⁷.

$$(17) \quad |B_k| \lesssim s_k \beta_k^{-(2k-2)} \prod_{j=1}^{k-1} \beta_j^2 \leq s_k \frac{\beta_{k-1}^2}{\beta_k^2} \leq s_k 5^{-2k}$$

We first estimate the terms in the sum and the sum itself. By (14), the sum $\sum_{k=1}^{\infty} |z|^2 |\beta_k|^{-2}$ converges and each infinite product in the sum converges ([31], see also the estimates below). It is clear that for a given $|z|$ all infinite products in (16) are maximal when $z = -|z|$.

We have

$$(18) \quad \sum_{k=1}^N B_k \prod_{j \neq k} \left(1 + \frac{\rho}{\beta_j}\right)^2 = \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{\beta_j}\right)^2 \sum_{k=1}^N \frac{B_k}{(1 + \rho/\beta_k)^2} \leq \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{\beta_j}\right)^2 \sum_{k=1}^N B_k$$

which converges by the assumption on β_k , (17), and the convergence of $\sum_{j=1}^{\infty} |z|^2 |\beta_k|^{-2}$.

We then also have

$$(19) \quad \begin{aligned} \sum_{k=1}^{\infty} B_k \prod_{j \neq k} \left(1 + \frac{\rho}{\beta_j}\right)^2 &\leq \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{\beta_j}\right)^2 \sum_{k=1}^{\infty} B_k \\ &\lesssim \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{\beta_j}\right)^2 = \prod_{\beta_j < \rho} \left(1 + \frac{\rho}{\beta_j}\right)^2 \prod_{\beta_j \geq \rho} \left(1 + \frac{\rho}{\beta_j}\right)^2 \lesssim 4^M \rho^{2M} \leq (4\rho^2)^{G^{-1}(\rho)} \end{aligned}$$

where M is the largest j s.t. $\beta_j < \rho$ for $j \leq M$ and we used $(1+x)^2 < 4x^2$ for $x > 1$. The inequality implies, in particular that f is entire.

⁷ For $x, y \geq 0$, the notation $x \lesssim y$ means $x \leq Cy$, where $C \geq 0$ does not depend on x, y , or relevant parameters. This notation is standard.

To estimate f_a , for $|z| \leq \rho$ we simply use Cauchy's formula on a circle of radius 2ρ :

$$(20) \quad |F'_a(z)| = \left| \frac{1}{2\pi i} \oint_{|s|=2\rho} \frac{F_a(s)}{(s-z)^2} \right| \lesssim \rho^{2g(\rho)-1} = W(\rho) \Rightarrow f_a(1/t)t^{-2} \in w^{\mathfrak{N}}$$

Since $F_a(\beta_k) = s_k$, with $f_1(t) = f(1/t)t^{-2}$ and $\alpha_k := 1/\beta_k$ we get

$$(21) \quad \forall k \in \mathbb{Z}^+ \Rightarrow \int_{\alpha_k}^{\alpha_{k+1}} f_1 = (s_{k+1} - s_k) = a_k$$

the desired result. \square

ACKNOWLEDGMENTS

This research was partially supported by NSF DMS Grant 1108794 (OC), an Ohio State University Presidential Research Grant and by the John Templeton Foundation⁸ grant ID #36297 (HF). The authors are grateful to A. Kechris for his very useful suggestions.

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⁸The opinions expressed here are those of the author and do not necessarily reflect the views of the John Templeton Foundation

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